FREE TRIAL

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— EDUCATION —

Write your **full name** in the space above.

Mathematical Methods Examination 2

Question and Answer Book

VCE Trial Examination – Free

- Reading time is **15 minutes**
- Writing time is **2 hours**

Approved materials

- Protractors, set squares and aids for curve sketching
- One bound reference
- One approved CAS calculator or CAS software, and one scientific calculator

Materials supplied

- Question and Answer Book of 24 pages
- Formula Sheet
- Multiple-Choice Answer Sheet

Instructions

- Follow the instructions on your Multiple-Choice Answer Sheet.
- At the end of the examination, place your Multiple-Choice Answer Sheet inside the front cover of this book.

Students are **not** permitted to bring mobile phones and/or any unauthorised electronic devices into the examination room.

Contents pages

Section A – Multiple-choice questions

Instructions

- Answer **all** questions in pencil on your Multiple-Choice Answer Sheet.
- Choose the response that is **correct** for the question.
- A correct answer scores 1; an incorrect answer scores 0.
- Marks will **not** be deducted for incorrect answers.
- No marks will be given if more than one answer is completed for any question.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The range of the function $f(x) = 2\sin(x) - 3$ is

A. $[-1, 5]$

B. $[-5, -1]$

C. $[-2, 2]$

D. $[1, 5]$

Question 2

The distance between the points $(1,3)$ and $(-2,6)$ is

A. $\sqrt{5}$

B. $\sqrt{10}$

C. $3\sqrt{2}$

D. $5\sqrt{2}$

Question 3

The angle measured anticlockwise from the positive direction of the x-axis to the line $x + \sqrt{3}y = 1$ is

A. $\frac{\pi}{6}$

B. $\frac{\pi}{3}$

C. $\frac{2\pi}{3}$

D. $\frac{5\pi}{6}$

Question 4

A box contains four red marbles and six blue marbles. Two marbles are drawn at random from the box without replacement.

The probability that the marbles are the same colour is

A. $\frac{7}{15}$

B. $\frac{8}{15}$

C. $\frac{12}{25}$

D. $\frac{13}{25}$

Question 5

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions such that $f(x) = 1 - 4x$ and $g(2x+1) = x^2 - x - 1$.

The rule of $f \circ g$ is given by

A. $(f \circ g)(x) = -x^2 + 4x + 2$

B. $(f \circ g)(x) = -4x^2 + 4x + 5$

C. $(f \circ g)(x) = 4x^2 + 2x - 1$

D. $(f \circ g)(x) = 16x^2 - 4x - 1$

Question 6

Consider the system of equations

$$\begin{aligned}(m+1)x + \ 3y &= 10 \\ 4x + my &= 4\end{aligned}$$

where $x, y \in \mathbb{R}$ and m is a real constant.

The system of equations has no solutions if and only if

A. $m = -4$

B. $m = 3$

C. $m \in \{-4, 3\}$

D. $m \in \mathbb{R} \setminus \{-4, 3\}$

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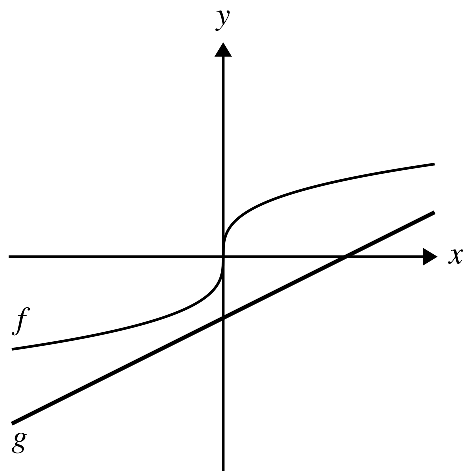
Question 7

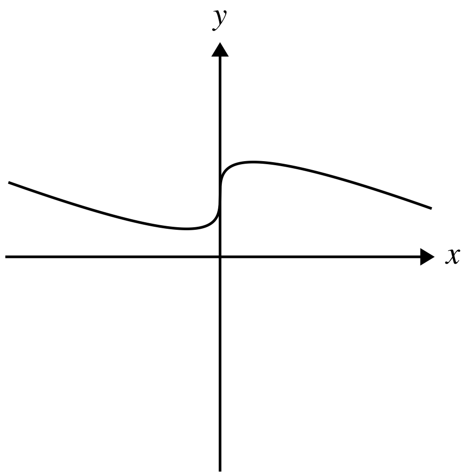
The graphs of the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are shown below.



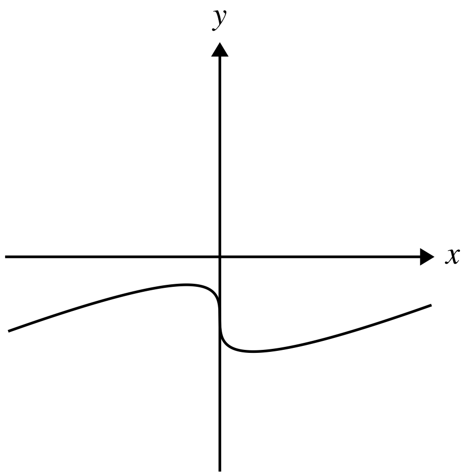
All axes below have the same scale as those in the diagram above.

Which one of the following options best represents the graph of the function $f - g$?

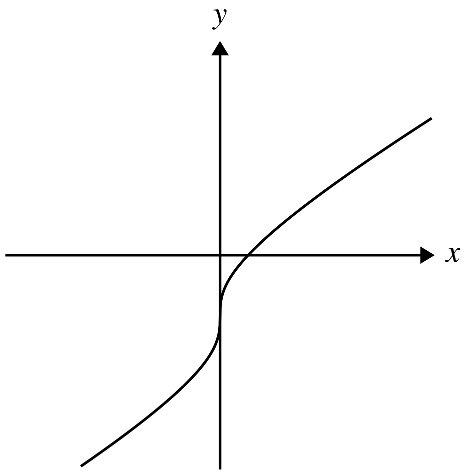
A.



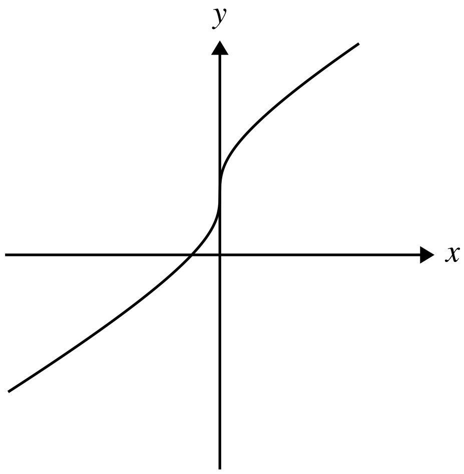
B.



C.



D.



Question 8

Let X be a discrete random variable defined by the probability distribution below.

x	-1	0	1	2
$\Pr(X = x)$	0.2	0.25	0.4	0.15

The variance of X is equal to

A. 0.5

B. 0.95

C. 1

D. 1.2

Question 9

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\int_0^2 f(x)\,dx = 4$ and $\int_0^4 f(x)\,dx = 10$.

Then $\int_2^4 \left(3f(x) - x\right) dx$ is equal to

A. 6

B. 8

C. 12

D. 36

Question 10

Let A and B be independent events of a sample space Ω such that $\Pr(A) = \frac{1}{3}$ and $\Pr(B) = \frac{1}{2}$.

Then $\Pr(A' \cap B')$ is equal to

A. $\frac{1}{6}$

B. $\frac{1}{3}$

C. $\frac{2}{3}$

D. $\frac{5}{6}$

Do not write in this area.

Question 11

Which one of the following functional equations does $f(x) = \dfrac{e^x + e^{-x}}{2}$ satisfy?

A. $f(2x) = 2\left(f(x)\right)^2 - 1$

B. $f(2x) = 2\left(f(x)\right)^2 + 1$

C. $f(2x) = \left(f(x)\right)^2 - 1$

D. $f(2x) = \left(f(x)\right)^2 + 1$

Question 12

The graph of $y = 4\cos\left(\dfrac{x}{2}\right) - x$ has points of inflection where

A. $x = k\pi,\ k \in \mathbb{Z}$

B. $x = 2k\pi,\ k \in \mathbb{Z}$

C. $x = (2k-1)\pi,\ k \in \mathbb{Z}$

D. $x = 2(2k-1)\pi,\ k \in \mathbb{Z}$

Question 13

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $f(2) = -3$.

Then $f(5)$ is equal to

A. $\displaystyle\int_{-3}^{5} \left(f'(x) + 2\right) dx$

B. $\displaystyle\int_{2}^{5} \left(f'(x) - 3\right) dx$

C. $\displaystyle\int_{-3}^{5} f'(x)\, dx + 2$

D. $\displaystyle\int_{2}^{5} f'(x)\, dx - 3$

Question 14

Let X be a discrete random variable with probability mass function

$$\Pr(X = x) = \frac{2^x}{e^2 x!}, \quad x \in \{0, 1, 2, ...\}.$$

Then $\Pr(X > 1)$ is closest to

A. 0.1353

B. 0.4060

C. 0.5940

D. 0.8647

Question 15

Consider the cubic polynomial $p(x) = x^3 + bx^2 + cx$, where b and c are real constants.

The graph of p has two stationary points if and only if

A. $c < \dfrac{b^2}{3}$

B. $c > \dfrac{b^2}{3}$

C. $c < \dfrac{b^2}{4}$

D. $c > \dfrac{b^2}{4}$

Question 16

Consider the following pseudocode.

```
define f(x):
    return x^3 + x + 1

define f_prime(x):
    return 3*x^2 + 1

x ← -1

while true do
    x_new ← x - f(x) / f_prime(x)

    if -0.001 < x_new - x < 0.001 then
        break
    end if

    x ← x_new
end while

print x_new
```

The number of times the **while** loop will run before the program terminates is

A. 3

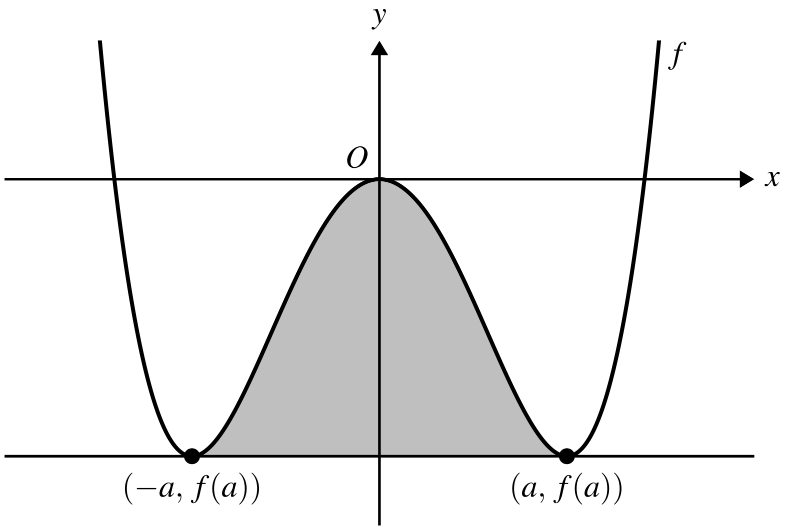
B. 4

C. 5

D. 6

Question 17

The graph of an even function $f : \mathbb{R} \to \mathbb{R}$ is shown below. The horizontal line $y = f(a)$ is tangent to the graph of f at the points $(a, f(a))$ and $(-a, f(a))$, where $a > 0$.



The area of the shaded region enclosed by the graph of f and the line $y = f(a)$ is given by

A. $af(a) - \int_{-a}^{a} f(x)\,dx$

B. $2af(a) - 2\int_{0}^{a} f(x)\,dx$

C. $\int_{-a}^{a} f(x)\,dx - af(a)$

D. $2\int_{0}^{a} f(x)\,dx - 2af(a)$

Question 18

The period of the function $h(x) = \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{3}\right)$ is

A. 2π

B. 4π

C. 6π

D. 12π

Question 19

Let $X \sim \text{Bi}(n, p)$, where n is an integer greater than 1 and $p \in (0,1)$.

Given that $\Pr(X \geq 1) > 0.9$, it follows that

A. $n < \dfrac{\log_e(0.1)}{\log_e(1-p)}$

B. $n > \dfrac{\log_e(0.1)}{\log_e(1-p)}$

C. $n < \dfrac{\log_e(0.9)}{\log_e(1-p)}$

D. $n > \dfrac{\log_e(0.9)}{\log_e(1-p)}$

Question 20

Let $f : \mathbb{R} \to \mathbb{R}$ be an invertible continuously differentiable function such that the following hold.

$$f(-1) = 2, \quad f(2) = -3, \quad f'(-1) = -2, \quad f'(2) = -1$$

Then $(f^{-1})'(2)$ is equal to

A. $-\dfrac{1}{2}$

B. $-\dfrac{1}{3}$

C. $\dfrac{1}{3}$

D. $\dfrac{1}{2}$

Section B

Instructions

- Answer **all** questions in the spaces provided.
- Write your responses in English.
- In all questions where a numerical answer is required, an **exact** value must be given, unless otherwise specified.
- In questions where more than one mark is available, appropriate working **must** be shown.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (13 marks)

Consider the function $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2(x^2 - 8)$.

a. Find $f'(x)$. 1 mark

b. **i.** Find the minimum value of f and the value(s) of x for which it occurs. 2 marks

ii. State the coordinates of the local maximum of the graph of f. 1 mark

iii. Find all values of $p \in \mathbb{R}$ such that $f(x) + p = 0$ has exactly two solutions for x. 1 mark

c. Find the total area of the regions enclosed by the graph of f and the x-axis. 2 marks

Consider the function $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^4 + \frac{a}{6}x^3 - 8x^2 - 2ax$, where a is a real constant.

d. State the value of a for which $f(x) = g(x)$ for all $x \in \mathbb{R}$. 1 mark

e. Solve $g'(x) = 0$ for x, in terms of a where appropriate. 1 mark

f. **i.** State the minimum number of stationary points that the graph of g can have. 1 mark

ii. Find the value(s) of a for which the graph of g has a stationary point of inflection. 2 marks

iii. If the graph of g has a stationary point of inflection, state the range of g. 1 mark

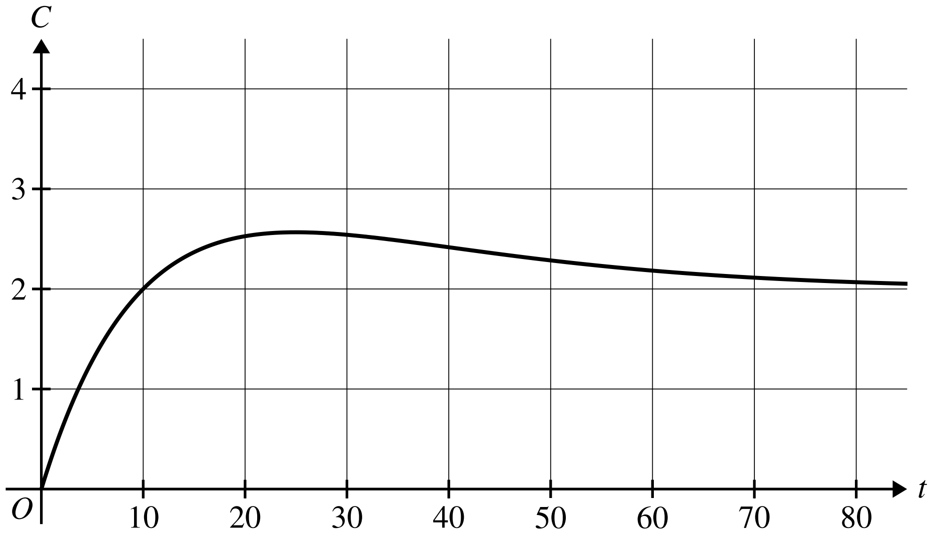
Question 2 (11 marks)

A pool cleaner system is used to dispense and mix chlorine into a pool containing pure water.

The concentration, C mg L^{-1}, of chlorine in the pool t minutes after the cleaning process begins is given by

$$C(t) = \frac{1}{5}(t-10)e^{-\frac{t}{15}} + 2$$

where $t \geq 0$. The graph of C is shown below.



a. Find the concentration of chlorine in the pool after 20 minutes. Give your answer in mg L^{-1}, correct to two decimal places. 1 mark

b. Find $C'(t)$ 1 mark

c. Find the largest interval on which C is strictly increasing. 1 mark

d. Find the maximum concentration of chlorine in the pool. Give your answer in mg L^{-1}, correct to two decimal places. 1 mark

e. Find the limiting concentration of chlorine in the pool as $t \to \infty$. Give your answer in mg L^{-1}. 1 mark

f. Find the average rate of change of C from $t = 0$ to $t = 25$. Give your answer in mg L^{-1} per minute, correct to three decimal places. 2 marks

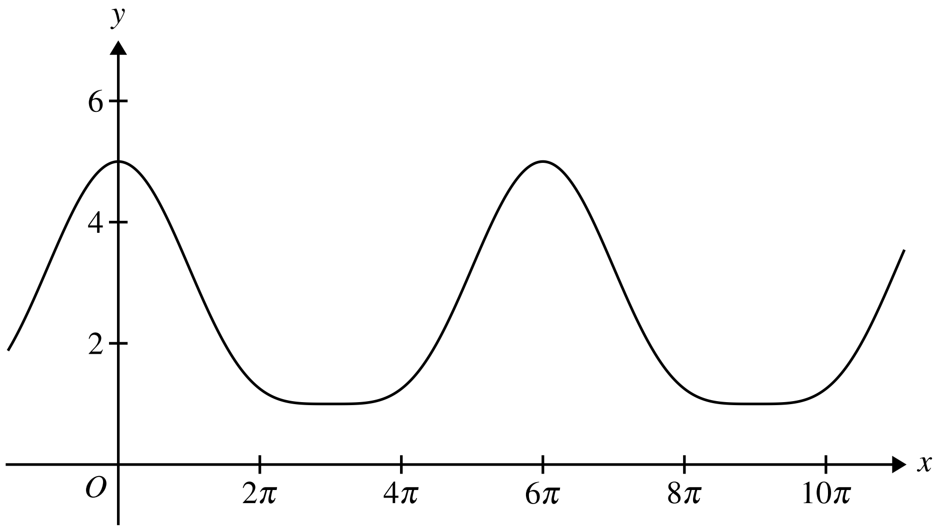
g. Find the average concentration of chlorine in the pool between $t = 0$ and $t = 25$. Give your answer in mg L^{-1}, correct to two decimal places. 2 marks

h. Find the value of t for which the concentration of chlorine in the pool is decreasing most rapidly. 2 marks

Do not write in this area.

Question 3 (10 marks)

The graph of $f : \mathbb{R} \to \mathbb{R},\ f(x) = \cos^2\left(\frac{x}{3}\right) + 2\cos\left(\frac{x}{3}\right) + 2$ is shown below.



a. State the period of f. 1 mark

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b. Find the set of values $x \in [0, 12\pi]$ for which $f(x) < 2$. Express your answer as a union of open intervals. 2 marks

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c. Consider the region bounded by the graph of f, the coordinate axes, and the line $x = 3\pi$.

Estimate the area of this region using the trapezium rule with a step size of $\frac{\pi}{2}$. 2 marks

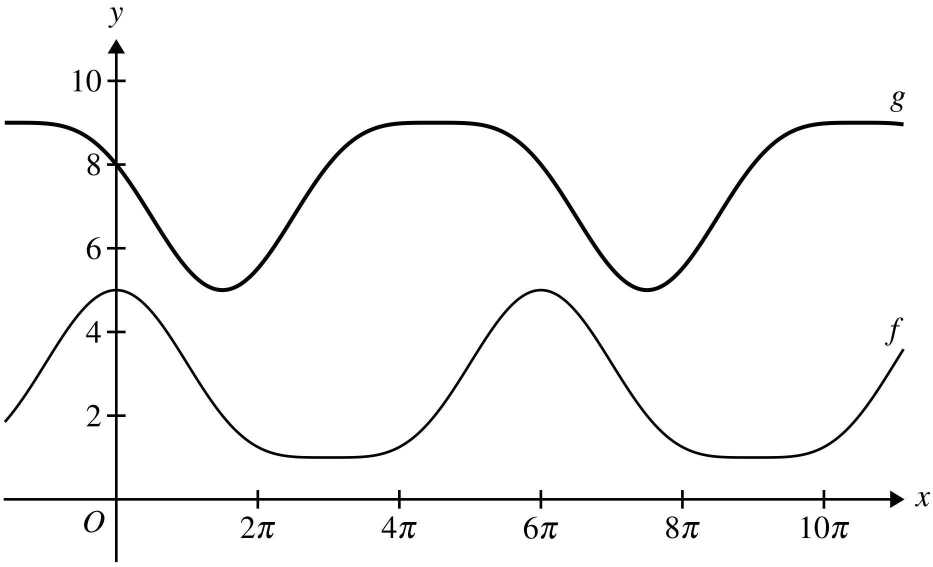
The graph of f is mapped to the graph of a function $g : \mathbb{R} \to \mathbb{R}$ by the following sequence of transformations:

1. reflection in the x-axis
2. translation of $\frac{3\pi}{2}$ units in the positive x-direction
3. translation of 10 units in the positive y-direction

d. Show that the rule of g is given by $g(x) = 8 - 2\sin\left(\frac{x}{3}\right) - \sin^2\left(\frac{x}{3}\right)$. 2 marks

Question 3 continues on the next page.

The graphs of f and g are shown below.



e. Find the maximum vertical distance between the graphs of f and g, and all the x-coordinates where this maximum occurs. 3 marks

Question 4 (15 marks)

The diameters of tennis balls produced by Factory A are known to be normally distributed with a mean of 67 mm and a standard deviation of 1.2 mm.

To meet official requirements, the diameter of a tennis ball must be between 65.4 mm and 68.6 mm.

a. A tennis ball produced by Factory A is randomly selected.

i. Find the probability that it meets the official diameter requirement, correct to four decimal places. 1 mark

ii. Given that the tennis ball meets the official diameter requirement, find the probability that its diameter is at least 68 mm. Give your answer correct to four decimal places. 2 marks

Factory A installs new production equipment. With the new equipment, the diameters of tennis balls remain normally distributed with a mean of 67 mm, and 95% of tennis balls produced meet the official diameter requirement.

b. Find the standard deviation of the diameters of tennis balls produced using the new equipment. Give your answer in millimetres, correct to two decimal places. 3 marks

Question 4 continues on the next page.

Factory A only produces tennis balls using the new equipment. The proportion of tennis balls that do **not** meet the official diameter requirement is now 0.05.

A random sample of 75 independently produced tennis balls is selected. For random samples of 75 tennis balls, let $\hat{P}$ be the random variable representing the proportion of tennis balls that do **not** meet the official diameter requirement.

c. Find $\Pr\left(\hat{P} > 0.1\right)$, correct to four decimal places. Do not use a normal approximation. 2 marks

d. **i.** Find the expected value and standard deviation of $\hat{P}$. 2 marks

ii. Find the probability that $\hat{P}$ lies within one standard deviation of its mean, correct to four decimal places. Do not use a normal approximation. 2 marks

e. Find the mode of $\hat{P}$; that is, the value of $k \in \left\{0, \frac{1}{75}, \frac{2}{75}, \ldots, 1\right\}$ for which $\Pr\left(\hat{P} = k\right)$ is maximised. 1 mark

At Factory B, a random sample of 240 independently produced tennis balls is selected. Of these, 235 are found to meet the official diameter requirement.

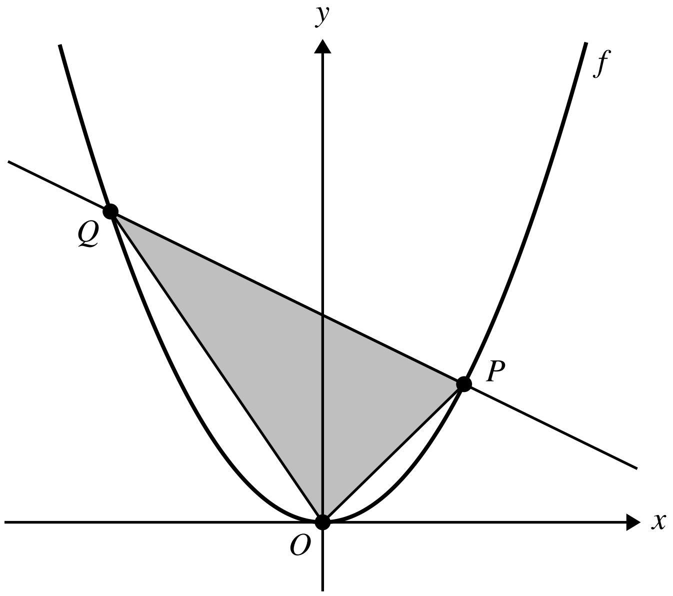
f. Find an approximate 95% confidence interval for the proportion of tennis balls produced by Factory B that meet the official diameter requirement. Give your answer correct to four decimal places. 1 mark

g. Does the confidence interval found in **part f.** suggest that the proportion of tennis balls produced by Factory B that meet the official diameter requirement is different to that of Factory A using the new equipment? Justify your answer. 1 mark

Question 5 (11 marks)

Consider the function $f : \mathbb{R} \to \mathbb{R},\ f(x) = \dfrac{x^2}{2}$.

The normal (line perpendicular) to the graph of f at $P(a, f(a))$, where $a > 0$, intersects the graph of f again at Q. The points O, P and Q form a triangle, as shown below.



a. Find the equation of the normal to the graph of f at P, in terms of a. 1 mark

b. Find the coordinates of Q, in terms of a. 2 marks

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Let $A : (0, \infty) \to \mathbb{R}$ be the function which determines the area of the triangle OPQ.

c. Find the rule of A, in terms of a. 3 marks

d. Find the value of a for which A is minimum. 1 mark

e. Find the value of a for which the angle QOP is $90°$. 2 marks

Consider the functions $g : [0, \infty) \to \mathbb{R},\ g(x) = \sqrt{2x}$ and $h : [0, \infty) \to \mathbb{R},\ h(x) = -\sqrt{2x}$.

The normal to the graph of g at $R\,(b,\, g(b))$, where $b > 0$, intersects the graph of h at S, and the points O, R and S form a triangle.

f. Find the value of b for which the area of the triangle ORS is minimum. 2 marks

End of Question and Answer Book

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